Critical fluctuations, intermittent dynamics and Tsallis statistics

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Abstract

It is pointed out that the dynamics of the order parameter at a thermal critical point obeys the precepts of the nonextensive Tsallis statistics. We arrive at this conclusion by putting together two well-defined statistical-mechanical developments. The first is that critical fluctuations are correctly described by the dynamics of an intermittent nonlinear map. The second is that intermittency in the neighborhood of a tangent bifurcation in such map rigorously obeys nonextensive statistics. We comment on the implications of this result.

Key words: critical fluctuations, intermittency, nonextensive statistics, anomalous stationary states

PACS: 64.60.Ht, 75.10.Hk, 05.45.-a, 05.10.Cc

1 Introduction

The modern theory of critical phenomena [1]-[4] has a distinguished history [4] of achievements in understanding the source of the scaling and universality associated to continuous phase transitions. The introduction of renormalization group (RG) concepts provided the means for calculating exponents,

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scaling functions and marginal dimensionalities of critical points in thermal and other statistical-mechanical models. Moreover, the RG methods have been fruitful in other areas of physics, in condensed matter problems, in non-linear dynamics and in other fields [4]. The success of the RG strategy in handling problems involving many length scales is illustrated by the use of a coarse-grained free energy or effective action. For instance, the equilibrium configurations of Ising spins at the critical temperature shows magnetic domains on all size scales and these are suitably studied by means of the Landau-Ginzburg-Wilson (LGW) continuous spin model [1]-[4].

Here we may add yet other universal aspect to critical point properties, the nonextensivity of order-parameter fluctuations (as explained in more detail below and in Ref. [5]). This previously unidentified property develops as the size of subsystems or domains of a thermal system at its critical point is allowed to become infinitely large. These domains have been shown [6]-[9], via the use of the LGW free energy, to possess intermittency properties, and these, in turn, are seen to comply [12] [13] with the presumptions of the nonextensive generalization [10] [11] of the Boltzmann-Gibbs (BG) statistical mechanics.

In a series of papers [6]-[9] a bridge has been built connecting the equilibrium dynamics of fluctuations at an ordinary thermal critical point with the intermittent dynamics of critical nonlinear maps at a tangent bifurcation. With the initial aim of investigating the origin of the relationship between the fractal geometry of clusters of the order parameter and the critical exponents of the phase transition, a theoretical approach was devised that lead to the evaluation of the partition function of a cluster or domain where the dominant contributions arise from a singularity (similar to an instanton) located in the space outside it [6] [7]. Subsequently [8] [9], a nonlinear map for the average order parameter was constructed whose dynamics reproduce the averages of the thermal critical properties. This map has as a main feature the tangent bifurcation and so time evolution is intermittent.

Recently [12] [13], we have shown that the known exact *static* solution of the RG equations for the tangent bifurcation in nonlinear maps also describes the *dynamics* of iterates. The fixed-point expressions have the specific form that corresponds to the temporal evolution of ensembles of iterates prescribed by the nonextensive formalism. The proof rests on the derivation of the sensitivity to initial conditions ξ_t exclusively from RG procedures without approximations followed by comparison with the nonextensive ξ_t. The study of the intermittency transition has been expanded via detailed deriva-

tion of their q-generalized Lyapunov coefficients λ_q and interpretation of the different types of sensitivity ξ_t [13]. Likewise, the properties of the intricate trajectories at the edge of chaos in unimodal maps have been analytically obtained leading too to the determination of λ_q and interpretation of the dynamics at the strange attractor [14] [15].

We combine the results mentioned in the previous two paragraphs to point out the nonextensive nature of critical fluctuations. This conclusion relates to the quest of the physical circumstances for which BG statistics fails to be applicable and its nonextensive generalization might offer correct descriptions. These anomalous situations are signalled by the vanishing of the Lyapunov coefficients (a single coefficient $\lambda_1 = 0$ for one-dimensional maps) and exhibit nonergodicity or unusual phase space mixing [12]-[14] [16]. At an intermittency transition hindered or incomplete mixing in phase space arises from the special 'tangency' shape of the map at its origin. This has the effect of confining or expelling trajectories causing irregular phase-space sampling, in contrast to the thorough coverage in generic states with $\lambda_1 > 0$. The occurrence of anomalous nonextensive states appears to be related with a nonuniform convergence of limits, such as the thermodynamic and infinite time limits. Here we comment on the nonuniform convergence associated to the description of critical domains as obtained from the LGW free energy.

Below we expand our arguments.

2 Criticality and intermittency

We briefly recall basic elements and results of the approach in Refs. [6]-[9]. The starting point is the partition function of the d-dimensional system at criticality,

$$Z = \int D[\phi] \exp(-\Gamma_c[\phi]), \tag{1}$$

where

$$\Gamma_c[\phi] = g_1 \int_\Omega dV \left[\frac{1}{2} (\nabla \phi)^2 + g_2 \left| \phi \right|^{\delta+1} \right] \tag{2}$$

is the critical LGW free energy of a system of d-dimensional volume Ω, ϕ is the order parameter (e.g. magnetization per unit volume) and δ is the critical isotherm exponent. The partition function Z was evaluated for a subsystem of size $V \sim R^d$ and by approximating the path integral in Eq. (1) as a summation over the saddle-point configurations of $\Gamma_c[\phi]$ - an approximation

valid for $g_1 \gg 1$. Integration of the Euler-Lagrange equation associated to the saddle points of $\Gamma_c[\phi]$, and identification of dominant saddle points, leads to power-law magnetization profiles for (spherically symmetric) critical clusters $\phi(r)$, and, finally, to the evaluation of the free energy $\Gamma_c[\phi]$ and the partition function Z in closed form [6] [7]. In doing this it was important to notice that only configurations with $r_0 \gg R$, where r_0 is a system-dependent reference position $r_0 = r_0(g_2, \delta, \phi(0))$, have a nonvanishing contribution to the path integration [6] [7]. These configurations vanish for the infinite size system, so there is nonuniform convergence in relation to the limits $R \to \infty$ and $r_0 \to \infty$, a feature that is significant for our connection with q-statistics. Based on the above results the fractal geometry of the critical clusters was determined and its relationship with the exponent δ was derived. The power-law dependence of the magnetization on the cluster radius was identified with the fractal dimension of the geometry of the cluster [6] [7]. Multifractal properties appear when global, rather than a single cluster, properties are considered.

Subsequent to this development, a link was revealed [8] [9] between the fluctuation properties of a critical system described by Eq. (1) and the dynamics of marginally chaotic intermittent maps. By considering the space-averaged magnetization $\Phi = \int_V \phi(x) dV$, the statistical weight

$$\rho(\Phi) = \exp(-\Gamma_c[\Phi])/Z, \tag{3}$$

where $\Gamma_c[\Phi] \sim g_1 g_2 \Phi^{\delta+1}$ and $Z = \int d\Phi \exp(-\Gamma_c[\Phi])$, was seen to be the invariant density of a statistically equivalent one-dimensional map. The functional form of this map was obtained as the solution of an inverse Frobenius-Perron problem [8] [17]. For small values of Φ the map has the form

$$\Phi_{n+1} = \Phi_n + u\Phi_n^{\delta+1} + \epsilon, \tag{4}$$

where the amplitude u depends on g_1, $g_2$and δ, and the shift parameter $\epsilon \sim R^{-d}$. Eq. (4) can be recognized as that describing the intermittency route to chaos in the vicinity of a tangent bifurcation [18]. The complete form of the map displays a superexponentially decreasing region that takes back the iterate close to the origin in one step. Thus the parameters of the thermal system determine the dynamics of the map. Averages made of order-parameter critical configurations are equivalent to iteration time averages along the trajectories of the map close to the tangent bifurcation. The mean number of iterations in the laminar region was seen to be related

to the mean magnetization within a critical cluster of radius R. There is a corresponding power law dependence of the duration of the laminar region on the shift parameter ϵ of the map [8]. For $\epsilon > 0$ the (small) Lyapunov exponent is simply related to the critical exponent δ [7].

3 Intermittency and nonextensivity

Next, in a few words, we recall the nonextensive nature of the nonlinear dynamics at the tangent bifurcation [12] [13]. At this transition, the intermittency route to chaos, the ordinary Lyapunov exponent λ_1 vanishes and the sensitivity to initial conditions $\xi_t \equiv |dx_t/dx_0|$ (where x_t is the orbit position at time t given the initial position x_0 at time $t = 0$) is no longer given by the BG law $\xi_t = \exp(\lambda_1 t)$ but acquires either a power or a super-exponential law [13]. The nonextensive formalism indicates that ξ_t is given by the q-exponential expression,

$$\xi_t = \exp_q(\lambda_q t) \equiv [1 - (q-1)\lambda_q t]^{-1/(q-1)}, \tag{5}$$

containing the entropic index q and the q-generalized Lyapunov coefficient λ_q. Also, according to the generalized theory, the q-Pesin identity $K_q = \lambda_q$ replaces the ordinary Pesin identity $K_1 = \lambda_1$, $\lambda_1 > 0$, where $K_q \equiv t^{-1}\left[S_q(0) - S_q(t)\right]$ and $K_1 \equiv t^{-1}[S_1(t) - S_1(0)]$ are the rates of entropy increment based on the Tsallis entropy

$$S_q \equiv \sum_i p_i \ln_q \left(\frac{1}{p_i}\right) = \frac{1 - \sum_i^W p_i^q}{q-1}, \tag{6}$$

(where $\ln_q y \equiv (y^{1-q} - 1)/(1-q)$ is the inverse of $\exp_q(y)$) and the BG entropy

$$S_1(t) = -\sum_{i=1}^{W} p_i(t) \ln p_i(t), \tag{7}$$

respectively. (Above, $p_i(t)$ is the distribution obtained from the relative frequencies with which the positions of an ensemble of trajectories occur within cells $i = 1, ..., W$ at iteration time t). In the limit $q \to 1$ the expressions for the nonextensive theory reduce to the ordinary BG expressions. See Ref. [13] for a more rigorous description of the Pesin identity and related issues.

Assisted by the known RG treatment for the tangent bifurcation [18], the formula for ξ_t has been rigorously derived [12] [13] and found to comply with

Eq. (5). Also the validity of the q-Pesin identity has been substantiated for this problem [12]. The tangent bifurcation is usually studied by means of the map

$$f(x) = \epsilon + x + u\,|x|^{z} + O(|x|^{z}),\ u > 0, \tag{8}$$

in the limit $\epsilon \to 0$. The associated RG fixed-point map $x' = f^{*}(x)$ was found to be

$$x' = x\exp_{z}(ux^{z-1}) = x[1-(z-1)ux^{z-1}]^{-1/(z-1)}, \epsilon = 0, \tag{9}$$

as it satisfies $f^{*}(f^{*}(x)) = \alpha^{-1}f^{*}(\alpha x)$ with $\alpha = 2^{1/(z-1)}$ and has a power-series expansion in x that coincides with Eq. (8) in the two lowest-order terms. (Above $x^{z-1} \equiv |x|^{z-1}\operatorname{sgn}(x)$). The long time dynamics is readily derived from the static solution Eq. (9), one obtains

$$\xi_{t}(x_{0}) = [1-(z-1)ax_{0}^{z-1}t]^{-z/(z-1)}, \tag{10}$$

and so, $q = 2 - z^{-1}$ and $\lambda_{q}(x_{0}) = zax_{0}^{z-1}$ [12] [13]. When $q > 1$ the left-hand side ($x < 0$) of the tangent bifurcation map Eq. (8) exhibits a weak insensitivity to initial conditions, i.e. power-law convergence of orbits. However at the right-hand side ($x > 0$) of the bifurcation the argument of the q-exponential becomes positive and this results in a 'super-strong' sensitivity to initial conditions, i.e. a sensitivity that is faster than exponential [13].

4 Nonextensivity and criticality

The implications of joining the results described in the previous two sections are apparent. In the subsystem of infinite size $R \to \infty$ the dynamics of critical fluctuations obey the nonextensive statistics. This is expressed via the time series of the average order parameter Φ_{n}, i.e. trajectories Φ_{n} with close initial values separate in a superexponential fashion according to Eq. (10) with $q = (2\delta + 1)/(\delta + 1) > 1$ and with a q-Lyapunov coefficient λ_{q} determined by the system parameter values δ, g_{1}, g_{2} and Φ_{0} [5]. Also, when considering an ensemble of trajectories $\{\Phi_{n}\}$ with prescribed distribution of initial conditions, the q-Pesin identity $K_{q} = \lambda_{q}$ holds with the rate of entropy production K_{q} evaluated according to the nonextensive entropy S_{q}.

It is interesting to comment on the conditions for the incidence of q-statistical properties at criticality and the manner in which these develop. The order parameter profile for a large but finite-size domain $R >> 1$ has the form [6]-[8]

$$\phi(r) = A_{d}(r^{2} - r_{0}^{2})^{(2-d)/2}, d \geq 3, \tag{11}$$

that, parenthetically, can be rewritten in terms of a q-exponential. There is a singularity in $\phi(r)$ when $r = r_0$, the reference position. We keep in mind the requirement $R \ll r_0$ for the critical clusters $\phi(r)$ to be of relevance to the partition function Z and also the map shift parameter dependence on the domain size $\epsilon \sim R^{-d}$. The time evolution of Φ displays laminar episodes of duration $< n >\sim \epsilon^{-\delta/(\delta+1)}$ and the Lyapunov coefficient in this regime is $\lambda_1 \sim \epsilon$ [7]. Within the first laminar episode the dynamical evolution of Φ obeys q-statistics, but for very large times the occurrence of many different laminar episodes leads to an increasingly chaotic orbit consistent with the small $\lambda_1 > 0$ and BG statistics is recovered. As R increases ($R \ll r_0$ always) the time duration of the nonextensive regime increases and in the limit $R \to \infty$ there is only one infinitely long laminar nonextensive episode with $\lambda_1 = 0$ and with no crossover to BG statistics. On the other hand when $R > r_0$ the clusters $\phi(r)$ are no longer dominant, for the infinite subsystem $R \to \infty$ their contribution to Z vanishes [6]-[8] and no departure from BG statistics is expected to occur.

This study is developed in Ref. [5].

5 Current questions on q-statistics

What is the significance of the connections we have presented? Are there other connections between critical phenomena and transitions to chaos? Are all critical states - infinite correlation length with vanishing Lyapunov coefficients - outside BG statistics? When does BG statistics stop working? Where, and in that case why, does Tsallis statistics apply? Is ergodicity failure the basic playground for applicability of generalized statistics? Is there a link between critical dynamics and glassy dynamics?

Suggestive results and partial answers to these questions are given here and in the cited references. In relation to this it is pertinent to mention the following recent development. In Ref. [19] it is argued that the dynamics at the noise-perturbed edge of chaos in logistic maps is analogous to that observed in supercooled liquids close to vitrification. That is, the three major features of glassy dynamics in structural glass formers, two-step relaxation, aging, and a relationship between relaxation time and configurational entropy, are displayed by orbits with $\lambda_1 = 0$. Time evolution is obtained from the Feigenbaum RG transformation, it is expressed analytically via q-exponentials, and described in terms of nonextensive statistics.

Acknowledgments. It is a great pleasure to dedicate this work to Constantino Tsallis on the occasion of his 60th birthday. I am grateful for hospitality offered at Angra dos Reis, Brazil, and for partial support by DGAPA-UNAM and CONACyT (Mexican Agencies).

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